

SCHLICHT SOLUTIONS OF $W'' + pW = 0$

BY

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1. Introduction. Let

$$(1.1) \quad z^2 p(z) = p_0 + p_1 z + \cdots + p_n z^n + \cdots$$

be regular for $|z| < 1$. For the differential equation

$$(1.2) \quad \frac{d^2 W}{dz^2} + p(z)W = 0,$$

the origin is a regular singular point (or an ordinary point when p_0 and p_1 are both zero), and the indicial equation is

$$(1.3) \quad \lambda^2 - \lambda + p_0 = 0,$$

with roots α and β , $\alpha + \beta = 1$, $\Re \alpha \geq 1/2 \geq \Re \beta$. Corresponding to the root α with the larger real part (or to either root if the real parts are equal), there exists a unique solution of the form

$$(1.4) \quad W = W(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1,$$

valid in the unit circle $|z| < 1$. Let $F(z)$ be defined as

$$(1.5) \quad F(z) = \{W(z)\}^{1/\alpha} = z + \cdots,$$

where that branch of the function is chosen for which $F'(0) = 1$. In this paper we shall obtain sufficient conditions on $p(z)$, of a fairly general nature, so that $F(z)$ is schlicht in $|z| < 1$ ($F(z)$ takes on no value more than once in the unit circle).

It will be observed immediately that every analytic function $f(z)$, schlicht in $|z| < 1$,

$$(1.6) \quad f(z) = z + \mu_2 z^2 + \cdots + \mu_n z^n + \cdots,$$

satisfies an equation of the form (1.2) where

$$(1.7) \quad zp(z) = \frac{-zf''(z)}{f(z)}$$

is regular for $|z| < 1$. In this case, $p_0 = 0$ and $\alpha = 1$. It follows then, for the special instance $\alpha = 1$, that our sufficiency conditions on $p(z)$ can be re-phrased in terms of $f(z)$ and $f''(z)$ to give many new tests for an arbitrary analytic function $f(z)$ for which $f(0) = 0$, $f'(0) = 1$, to be schlicht in $|z| < 1$.

Presented to the Society, September 3, 1953; received by the editors February 28, 1953.

A related problem was recently solved by Nehari [5], who showed that in the special case $p_0 = p_1 = 0$, $\alpha = 1$, no solution of (1.2) can take on the value zero more than once in $|z| < 1$, provided either

$$(i) \quad |p(z)| \leq (1 - |z|^2)^{-2}, \quad |z| < 1,$$

or

$$(ii) \quad |p(z)| \leq \pi^2/4, \quad |z| < 1.$$

In either of these two cases the ratio of two independent solutions of (1.2) is schlicht in $|z| < 1$. The conditions (i) and (ii) may be replaced by

$$|p(z)| \leq \begin{cases} 2^{3\lambda-2} \pi^{2(1-\lambda)} (1 - |z|^2)^{-\lambda}, & \text{for } 0 \leq \lambda \leq 1, \\ 2^{2-\lambda} (1 - |z|^2)^{-\lambda}, & \text{for } 1 \leq \lambda \leq 2, \end{cases}$$

a refinement of Nehari's result due to Pokornyi [6]. A condition analogous to (ii) applied to analytic functions in a convex domain was obtained recently by Ryll-Nardzewski [8]. The problem at hand now, however, is to find sufficient conditions on $p(z)$ so that certain individual solutions can take on no value more than once in $|z| < 1$ (even when p_1 is different from zero). The "Green's Transform" of (1.2), used so successfully by Nehari [5] and of fundamental importance in the earlier papers of Hille [1; 2; 4] on the existence of zero-free regions for solutions of (1.2), also plays an important role in this investigation.

Our aim is to derive a fairly general "parent" theorem, involving a somewhat arbitrary function $p^*(z)$, whose "offspring" will be theorems corresponding to each selected function $p^*(z)$. Each such $p^*(z)$ will have associated with it a universal constant $A = A(p^*)$ (often times a root of a transcendental equation) which will give a sharp character to the corresponding theorem. By varying $p^*(z)$ innumerable tests for the univalence of $F(z)$ of (1.5) may be obtained. A few of these examples will be explored in §7 of this paper. Because of the length of a satisfactory and complete statement of the main theorem, we postpone this until the proof is at hand in §6.

2. Preliminary definitions. Let

$$(2.1) \quad f(z) = z + \mu_2 z^2 + \cdots + \mu_n z^n + \cdots$$

be regular for $|z| < 1$. We denote by S the class of functions $f(z)$, $f(0) = 0$, $f'(0) = 1$, given by (2.1), which are schlicht, or univalent, in $|z| < 1$. Let $S(\gamma)$ be the subclass of S whose members $f(z)$ satisfy, for some real constant γ ($|\gamma| \leq \pi/2$) and $|z| < 1$, the inequality

$$(2.2) \quad \Re \left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} \geq 0.$$

It was shown by Špaček [9] that the inequality (2.2) is a sufficient condition

(when $f'(0) \neq 0$) for $f(z)$ of (2.1) to be schlicht in $|z| < 1$. In general, a member of $S(\gamma)$ maps $|z| < 1$ onto a spiral-like domain. We shall call $f(z)$ "spiral-like" if it is a member of $S(\gamma)$.

The subclass $S(0)$ of $S(\gamma)$, corresponding to $\gamma = 0$, contains only those members of S which are star-like with respect to the origin. Each member $f(z)$ of $S(0)$ maps $|z| < 1$ onto a star-domain, one which has the property that every ray from the origin contains an open segment, from the origin to a boundary point, which lies entirely within the domain. We shall call $f(z)$ "star-like" if it is a member of $S(0)$. Every star-like function is also spiral-like since $S(0) \subset S(\gamma)$.

We shall denote by K the subclass of $S(0)$ whose members $f(z)$ map $|z| < 1$ onto a convex domain, one which has the property that, if W_1 and W_2 are points within the domain, the line segment joining W_1 and W_2 lies entirely within the domain. It is well known that $f(z)$, with $f(0) = 0$ and $f'(0) = 1$, is convex in $|z| < 1$ if, and only if, $zf''(z)$ is star-like in $|z| < 1$. Since (2.2), with $\gamma = 0$, is both necessary and sufficient for $f(z)$ to be star-like in $|z| < 1$, it follows (as is well known) that the necessary and sufficient condition that $f(z)$ (with again $f(0) = 0$, $f'(0) = 1$) be convex in $|z| < 1$ is that

$$(2.3) \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{for } |z| < 1.$$

Another important subclass of S , to be denoted by K^* , is the class whose members $f(z)$ are real on the real axis and each $f(z)$ maps $|z| < 1$ onto a domain which is convex in the direction of the imaginary axis. This means that if W_1 is any point within the domain, then the line segment joining W_1 and its conjugate point $\overline{W}_1$ lies entirely within the domain. It is known that the necessary and sufficient condition for $f(z)$ to belong to K^* is that $zf''(z) = g(z)$ be typically-real for $|z| < 1$. Following Rogosinski [7] we say $g(z)$ ($g(0) = 0$, $g'(0) = 1$) is typically-real for $|z| < 1$ if $g(z)$ is regular in $|z| < 1$, is real on the real axis, and if the imaginary part of $g(z)$ vanishes for $|z| < 1$ only when z is real. We shall speak of $f(z)$ as being "convex in the direction of the imaginary axis" if it is a member of K^* .

Let $f(z)$ be a member of S , and let $Cf(z)$ (C any constant not zero) map $|z| < 1$ onto a simply-connected domain D . Let $h(z)$ be regular in $|z| < 1$, and $h(0) = 0$. We shall say that $h(z)$ is "subordinate" to $Cf(z)$ in $|z| < 1$, and write

$$h(z) \prec\prec Cf(z)$$

whenever $h(z)$ lies within D for all z , $|z| < 1$.

3. Green's transform. Following Hille [2], we adjoin to the differential equation

$$(3.1) \quad W'' + p(z)W = 0, \quad \left(W'' = \frac{d^2W}{dz^2} \right),$$

the "Green's transform" of (3.1) obtained as follows. We multiply (3.1) by $\overline{W}dz$ and integrate from 0 to z , $|z| < 1$. This gives

$$(3.2) \quad \int_0^z \overline{W}(z) W''(z) dz + \int_0^z p(z) |W(z)|^2 dz = 0,$$

where $p(z)$ is given as in (1.1) and $W = W(z)$ is the solution (1.4), $\Re \alpha \geq 1/2$. For the present we shall assume $\Re \alpha > 1/2$ (the case $\Re \alpha = 1/2$ will be considered separately). Since $\Re \alpha > 1/2$ the integrals appearing in (3.2) exist. Integrating by parts, we obtain

$$(3.3) \quad [\overline{W}(z) W'(z)]_0^z - \int_0^z |W'(z)|^2 dz + \int_0^z p(z) |W(z)|^2 dz = 0.$$

Let the path of integration in (3.3) be the straight line segment, $\theta = \text{constant}$, joining the origin to the point $z = re^{i\theta}$, $r < 1$. Now multiply (3.3) by $ze^{i\gamma}$ and equate the real part of the resulting equation to zero. We then have

$$(3.4) \quad |W(z)|^2 \Re \left\{ e^{i\gamma} \frac{z W'(z)}{W(z)} \right\} \\ = r \cos \gamma \int_0^r |W'|^2 d\rho - r \int_0^r \Re \{ e^{i\gamma} z^2 p(z) \}_{|z|=\rho} \frac{|W|^2 d\rho}{\rho^2},$$

where $\Re F(z)$ denotes the real part of $F(z)$. The Green's transform, written in the form (3.4), will be of fundamental importance in the paragraphs to follow.

4. A fundamental inequality of integrals. Let

$$(4.1) \quad z^2 p^*(z) = p_0^* + p_1^* z + \cdots + p_n^* z^n + \cdots$$

be regular for $|z| < 1$, real on the real axis, and $p_0^* \leq 1/4$. If C is any non-negative constant, the differential equation

$$(4.2) \quad \frac{d^2 W}{dz^2} + \left\{ C \left(p^*(z) - \frac{p_0^*}{z^2} \right) + \frac{p_0^*}{z^2} \right\} W = 0$$

has its indicial equation

$$(4.3) \quad \lambda^2 - \lambda + p_0^* = 0$$

independent of C . Let α^*, β^* be the roots of (4.3). Since $p_0^* \leq 1/4$, α^* and β^* are real and so we have $\alpha^* \geq 1/2 \geq \beta^*$, $\alpha^* + \beta^* = 1$, where

$$(4.4) \quad 2\alpha^* = 1 + (1 - 4p_0^*)^{1/2}, \quad 2\beta^* = 1 - (1 - 4p_0^*)^{1/2}.$$

Let

$$(4.5) \quad W_C = W_C(z) = z^{\alpha^*} \sum_{n=0}^{\infty} a_n^*(C) z^n, \quad a_0^*(C) = 1,$$

be the unique solution of (4.2), corresponding to the real root α^* of (4.3), and depending upon the non-negative parameter C . The coefficients $a_n^* = a_n^*(C)$, all real, are determined from the recurrence relation

$$(4.6) \quad a_n^* = - \frac{C[p_n^* + p_{n-1}^* a_1^* + \cdots + p_1^* a_{n-1}^*]}{n(n + 2\alpha^* - 1)}, \quad n \geq 1.$$

From (4.6) it is readily seen that $a_n^*(C)$ is a polynomial in C of degree not exceeding n , and, in particular, a continuous function of C for $\alpha^* > 0$.

We shall show now that $W_C(z)$ is continuous in each of the three variables C , α^* , and z for $C \geq 0$, $\alpha^* > 0$, $|z| < 1$. If all the coefficients p_k^* , $k = 1, 2, \dots$, in the expansion (4.1) are replaced by their absolute values, and if p_0^* is replaced by $\epsilon - \epsilon^2$, ϵ arbitrarily small and positive, we have a new function which we shall call $\tilde{p}(z)$ and

$$(4.7) \quad z^2 \tilde{p}(z) = \epsilon - \epsilon^2 + \sum_{k=1}^{\infty} |p_k^*| z^k.$$

Let $C_0 > 0$ be chosen arbitrarily large. Then the differential equation

$$(4.8) \quad \frac{d^2 W}{dz^2} + \left\{ -C_0 \left(\tilde{p}(z) - \frac{\epsilon - \epsilon^2}{z^2} \right) + \frac{\epsilon - \epsilon^2}{z^2} \right\} W = 0$$

has the unique solution

$$(4.9) \quad \tilde{W}_{C_0}(z) = z^\epsilon \sum_{n=0}^{\infty} A_n(C_0) z^n, \quad A_0(C_0) = 1,$$

corresponding to the smaller root $\epsilon > 0$ of the indicial equation. Since $z^2 \tilde{p}(z)$ is regular in $|z| < 1$ whenever $z^2 p^*(z)$ is, the series for $\tilde{W}_{C_0}(z)$ converges for $|z| < 1$ for all C_0 . Moreover, the coefficients $A_n(C_0)$ are determined by the recurrence relation

$$(4.10) \quad n(n + 2\epsilon - 1)A_n(C_0) = C_0[|p_n^*| + |p_{n-1}^*|A_1 + \cdots + |p_1^*|A_{n-1}].$$

For $0 \leq C \leq C_0$, $\alpha^* > \epsilon > 0$, it is readily seen that the coefficients $a_n^*(C)$ determined by (4.6) satisfy

$$(4.11) \quad n(n + 2\epsilon - 1)|a_n^*(C)| < C_0[|p_n^*| + |p_{n-1}^* a_1^*| + \cdots + |p_1^* a_{n-1}^*|],$$

$$(4.12) \quad |a_1^*(C)| < \frac{C_0 |p_1^*|}{2\epsilon} = A_1(C_0).$$

Thus

$$(4.13) \quad |a_n^*(C)| < A_n(C_0) \quad \text{for } n \geq 1, 0 \leq C \leq C_0, \alpha^* > \epsilon.$$

Since the series $\sum_{n=0}^{\infty} A_n(C_0) z^n$ converges uniformly in z for $|z| < R < 1$, it follows that the series $\sum_{n=0}^{\infty} a_n^*(C) z^n$ converges uniformly in the three vari-

ables C , α^* , and z for $0 \leq C \leq C_0$, $\alpha^* > \epsilon > 0$, and $|z| < R < 1$. A similar statement holds for the derived series $\sum_{n=0}^{\infty} (n + \alpha^*) a_n^*(C) z^n$ if α^* is bounded above. Since the coefficients $a_n^*(C)$ are continuous functions of C and α^* and the series $\sum_{n=0}^{\infty} a_n^*(C) z^n$ converges uniformly in C , α^* , and z , it follows that $W_C(z)$ and its derivatives with respect to z are continuous functions of C , α^* , and z for $C \geq 0$, $\alpha^* > 0$, $|z| < 1$. Similar arguments apply to $W(z)$ if $\Re \alpha > 0$ in (1.4).

Now that we are dealing with a function $W_C(z)$ continuous in C , and because, for $n \geq 1$, $a_n^*(0) = 0$, $a_0^*(0) = 1$, it is easily seen that

$$(4.14) \quad \lim_{C \rightarrow 0} W_C(z) = W_0(z) = z^{\alpha^*},$$

uniformly for $|z| \leq R$ for any positive $R < 1$, and $W_0(z)$ is the solution of (4.2) when $C = 0$. By z^α we shall mean $\exp(\alpha \log z)$, the principal branch of $\log z$ being chosen.

Although $W_C(z)$ is in general not single-valued in the neighborhood of the origin, the logarithmic derivative $W'_C(z)/W_C(z)$ is single-valued and has a simple pole at the origin. Since the coefficients are all real and $a_0^*(C) = 1$, $\alpha^* > 0$ in (4.5), it is seen that for each $C \geq 0$ we have

$$(4.15) \quad W'_C(\rho) > 0 \quad \text{for a range } 0 < \rho < r(C),$$

and for the same range at least we also have

$$(4.16) \quad W_C(\rho) = \int_0^\rho W'_C(\rho) d\rho > 0.$$

Thus

$$(4.17) \quad \frac{\rho W'_C(\rho)}{W_C(\rho)} > 0 \quad \text{for } 0 \leq \rho < r \leq 1,$$

where for C fixed, r is the smallest positive zero of $W'_C(\rho)$ (as a function of ρ) or one, whichever is smaller. By taking C sufficiently small we may obviously have r as near to one as we like. We shall see later that, under certain restrictions (not very severe) on $p^*(z)$, by taking $C = C(R)$ sufficiently large we can make $W'_C(R)$ vanish for any given $R < 1$.

We are now ready to prove the following inequality of integrals which is of fundamental importance in the proof of our main theorem. We state the inequality as a lemma.

LEMMA. Let $y(\rho)$, $dy(\rho)/d\rho = y'(\rho)$ be real functions, continuous in the real variable ρ for $0 < \rho < 1$. For small values of ρ let

$$y(\rho) = O(\rho^\delta), \quad y'(\rho) = O(\rho^{\delta-1}), \quad \delta > 1/2.$$

Then

$$(4.18) \quad \int_0^r \{C(\rho^2 p^*(\rho) - p_0^*) + p_0^*\} y^2(\rho) \frac{d\rho}{\rho^2} \\ \leq \int_0^r \{y'(\rho)\}^2 d\rho - \frac{W_c'(\tau)}{W_c(\tau)} \cdot y^2(\tau), \quad 0 < \tau < 1,$$

where $W_c(z)$ is the solution (4.5) of (4.2), and where $C(\geq 0)$ is chosen small enough so that (4.17) holds. Equality in (4.18) holds if, and only if, $y(\rho) = k W_c(\rho)$, $\alpha^* > 1/2$, where k is an arbitrary real constant.

The conditions $y(\rho) = O(\rho^\delta)$, $y'(\rho) = O(\rho^{\delta-1})$, $\delta > 1/2$ guarantee the existence of the integrals involved. The lemma is proved with the use of the following identity and partial integration.

$$(4.19) \quad \int_0^r \left[y'(\rho) - \frac{W_c'(\rho)}{W_c(\rho)} y(\rho) \right]^2 d\rho \\ = \int_0^r \{y'(\rho)\}^2 d\rho - \int_0^r 2y'(\rho)y(\rho) \frac{W_c'(\rho)}{W_c(\rho)} d\rho \\ + \int_0^r \left\{ \frac{W_c'(\rho)}{W_c(\rho)} \right\}^2 y^2(\rho) d\rho \\ = \int_0^r \{y'(\rho)\}^2 d\rho - \left[y^2(\rho) \frac{W_c'(\rho)}{W_c(\rho)} \right]_0^r \\ + \int_0^r y^2(\rho) \left[\frac{d}{d\rho} \left(\frac{W_c'}{W_c} \right) + \left(\frac{W_c'}{W_c} \right)^2 \right] d\rho \\ = \int_0^r \{y'(\rho)\}^2 d\rho - y^2(\tau) \frac{W_c'(\tau)}{W_c(\tau)} + \int_0^r \frac{W_c''(\rho)}{W_c(\rho)} y^2(\rho) d\rho \\ = \int_0^r \{y'(\rho)\}^2 d\rho - y^2(\tau) \frac{W_c'(\tau)}{W_c(\tau)} \\ - \int_0^r \{C(\rho^2 p^*(\rho) - p_0^*) + p_0^*\} y^2(\rho) \frac{d\rho}{\rho^2}.$$

Since the left-hand side of the identity (4.19) is non-negative, and zero only if $y(\rho) = k W_c(\rho)$, the inequality (4.18) follows. When equality exists in (4.18) it is necessary that $\alpha^* > 1/2$, $p_0^* < 1/4$. However, the inequality holds for $\alpha^* \geq 1/2$ if $\delta > 1/2$. This completes the proof of the lemma.

5. Some new universal constants. Let $z^2 p(z)$ be regular in $|z| < 1$ and be given as in (1.1). Let

$$(5.1) \quad W = W(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1, \quad |z| < 1,$$

be the solution (1.4) of the differential equation (1.2) associated with a given $p(z)$. We have a (1-1) correspondence between the function $p(z)$ of (1.1) and the solution $W(z)$ of (1.4). Similarly, we have a (1-1) correspondence between the function $p^*(z)$ of (4.1) and the associated solution $W_c(z)$ in (4.5) of the differential equation (4.2).

We shall now restrict $p(z)$ by making it satisfy an inequality involving $p^*(z)$ ($p^*(z)$ regarded as a given fixed function). Then we shall deduce an inequality involving the associated functions $W(z)$ and $W_c(z)$.

Let $C \geq 0$, γ ($|\gamma| \leq \pi/2$) be assigned constants. Let $p(z)$ be restricted so that

$$(5.2) \quad \Re\{e^{i\gamma} z^2 p(z)\} \leq \cos \gamma [C\{|z|^2 p^*(|z|) - p_0^*\} + p_0^*]$$

for $|z| < 1$, and let $\Re \alpha > 1/2$. Let C be chosen small enough so that (4.17) holds. Taking $z=0$, we note that (5.2) implies in particular that

$$(5.3) \quad \Re(e^{i\gamma} p_0) \leq p_0^* \cos \gamma \leq (1/4) \cos \gamma.$$

We prove now the following preliminary theorem, comparing the solutions $W(z)$ and $W_c(z)$.

THEOREM A. *Let $z^2 p(z)$ be regular in $|z| < 1$ and satisfy (5.2). Let the root α of (1.3) be the one for which $\Re \alpha \geq 1/2$. Let*

$$W(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1, \quad |z| < 1,$$

be the unique solution of (1.2) corresponding to α . Let $z^2 p^(z)$ be regular in $|z| < 1$ and real on the real axis with $\lim_{z \rightarrow 0} z^2 p^*(z) = p_0^* \leq 1/4$. Let*

$$W_c(z) = z^{\alpha^*} \sum_{n=0}^{\infty} a_n^*(C) z^n, \quad a_0^*(C) = 1,$$

be the solution of (4.2) where α^ is given by (4.4), $\alpha^* \geq 1/2$. Then*

$$(5.4) \quad \Re\left\{e^{i\gamma} \frac{z W'(z)}{W(z)}\right\} \geq \frac{|z| W_c'(|z|)}{W_c(|z|)} \cos \gamma \geq 0, \quad |z| \leq R < 1,$$

for those values C , $0 \leq C \leq C(R)$, for which

$$(5.5) \quad W_c'(r) > 0 \quad \text{in } 0 < r < R.$$

If we assume further that

$$(5.6) \quad \max_{|z|=r < 1} \Re\{z^2 p^*(z)\} = r^2 p^*(r),$$

then

$$(5.7) \quad \Re\left\{\frac{z W_c'(z)}{W_c(z)}\right\} \geq \frac{|z| W_c'(|z|)}{W_c(|z|)} > 0, \quad |z| < R < 1,$$

$0 \leq C \leq C(R)$, and $rW_c'(r)/W_c(r)$ is a nonincreasing positive function of r for $0 \leq r \leq R$ when $0 \leq C \leq C(R)$, and for $C = C(R)$ decreases from α^* to zero as r increases from 0 to $R < 1$.

To prove Theorem A we use the Green's transform as given in the form (3.4), and assume $\Re \alpha > 1/2$ to begin with. From (3.4), (5.2), and (4.18) of the fundamental lemma we have for $z = re^{i\theta}$, $r < 1$, θ constant,

$$\begin{aligned} & |W(z)|^2 \Re \left\{ e^{i\gamma} \frac{zW'(z)}{W(z)} \right\} \\ (5.8) \quad & \geq r \cos \gamma \int_0^r |W'|^2 d\rho - r \cos \gamma \int_0^r \frac{C\{\rho^2 p^*(\rho) - p_0^*\} + p_0^*}{\rho^2} |W|^2 d\rho \\ & \geq \frac{rW_c'(r)}{W_c(r)} \cos \gamma |W(z)|^2, \end{aligned}$$

which gives (5.4) when $\Re \alpha > 1/2$. But (5.4) holds also when $\Re \alpha = 1/2$ since, as we have seen previously, $zW'(z)/W(z)$ is a continuous function of α when $\Re \alpha > 0$.

In particular, if $p(z)$ is chosen so that

$$(5.9) \quad z^2 p(z) \equiv C\{z^2 p^*(z) - p_0^*\} + p_0^*,$$

and if $p^*(z)$ is chosen so that (5.6) holds, then it follows that the condition (5.2) is satisfied when $\gamma = 0$. Furthermore, the solution $W(z)$ of (1.2) given in (1.4) becomes identical with the solution $W_c(z)$ of (4.2) given in (4.5). Thus we may replace $W(z)$ by $W_c(z)$ in (5.4) when $\gamma = 0$ and obtain (5.7). Obviously, equality occurs in (5.7) when z is positive. Thus (5.7) shows that

$$\min_{|z|=r} \Re \frac{zW_c'(z)}{W_c(z)} = \frac{rW_c'(r)}{W_c(r)}.$$

Because the minimum of a harmonic function does not occur at an interior point of a domain, it follows that $rW_c'(r)/W_c(r)$ is a nonincreasing positive function of r for $0 \leq r \leq R < 1$. We shall see a little later that this function for $C = C(R)$ decreases from α^* to zero as r increases from 0 to $R < 1$.

We have seen that for a given $R < 1$ there exists a range for C , $0 \leq C \leq C(R)$, for which

$$(5.10) \quad W_c'(r) > 0 \quad \text{in } 0 < r < R.$$

We shall now show that (5.10) cannot hold for sufficiently large values of C whenever (5.6) holds, and when $z^2 p^*(z)$ is not identically a constant. Because of (5.7)

$$(5.11) \quad \phi(z) \equiv \frac{zW_c'(z)}{W_c(z)} = \alpha^* + b_1 z + \cdots + b_n z^n + \cdots$$

has a positive real part for $|z| < R < 1$. Hence the coefficients b_n satisfy the inequalities

$$(5.12) \quad |b_n| \leq \frac{2\alpha^*}{R^n}, \quad n = 1, 2, \dots$$

We conclude from (5.12) that $|b_n|$ is a bounded function of C . On the other hand b_n is a polynomial in C with coefficients which are functions of the coefficients p_k^* , $k \geq 1$, and $\alpha^* \geq 1/2$. For example,

$$b_1 = a_1^*(C) = -\frac{Cp_1^*}{2\alpha^*}, \quad b_2 = 2a_2^*(C) - (a_1^*(C))^2 = -\frac{[C^2p_1^{*2} + 4\alpha^2 Cp_2^*]}{4\alpha^2(2\alpha + 1)}.$$

It follows from this point of view that $|b_n|$ cannot be a bounded function of C unless $p_k^* = 0$ for $k \geq 1$. The apparent contradiction is eliminated only if either

(5.13) (i) $W'_C(r) > 0$ for all r in the interval $0 < r < R < 1$ whenever $0 \leq C \leq C(R) < \infty$, while at the same time $W'_C(r) < 0$ for some value of $r < R$ and for every C such that $C(R) < C < C(R) + \delta$, $\delta > 0$ arbitrarily small;

or

(5.14) (ii) $z^2 p^*(z)$ is identically a constant.

In the second case ($z^2 p^*(z) = \text{constant } p_0^*$) the solution $W_C(z)$ of (4.2) is the same as $W_0(z)$ ($C=0$) which we have seen in (4.14) to be z^{α^*} . For this function $W'_C(r) > 0$ for arbitrary $C \geq 0$ and all positive r . Thus $C(R) = \infty$. In all other cases $C(R)$ is finite. In what follows we shall suppose that this trivial case is ruled out.

We shall show now that for any fixed R in the range $0 < R < 1$

$$(5.15) \quad W'_{C(R)}(R) = 0.$$

Thus it is possible to determine the value of $C(R)$ by finding, for fixed R , the smallest positive root $C = C(R)$ of the equation $W'_C(R) = 0$.

To prove (5.15) we note by (5.13) that for each $\delta > 0$, and for some $r = r(\delta)$, $0 < r(\delta) < R$, we have $W'_{C(R)+\delta}\{r(\delta)\} \leq 0$. Let $\{\delta_n\}$ be a sequence of values of δ for which $\delta_n > 0$, $\lim_{n \rightarrow \infty} \delta_n = 0$, $\lim_{n \rightarrow \infty} r(\delta_n) = r_0$ exists. Then, obviously, $0 \leq r_0 \leq R$. We have already seen that $W'_C(r)$ is continuous in C and r . Consequently, since $W'_{C(R)+\delta_n}\{r(\delta_n)\} \leq 0$, we have in the limit as $\delta_n \rightarrow 0$ the inequality $W'_{C(R)}(r_0) \leq 0$. But $W'_{C(R)}(r) > 0$ for $0 < r < R$, so that in particular $W'_{C(R)}(r_0) \geq 0$. We must conclude, therefore, that not only does $W'_{C(R)}(r_0) = 0$, but $r_0 = 0$ or R . However, $\lim_{r \rightarrow 0} W'_{C(R)}(r)$ is never zero if $\alpha^* > 0$. This implies then that $r_0 \neq 0$. Thus $r_0 = R$ and $W'_{C(R)}(R) = 0$.

We note then by Theorem A and equality (5.15) that the function $[rW'_{C(R)}(r)/W_{C(R)}(r)]$ decreases from α^* to 0 as r increases from 0 to $R < 1$.

Since it is possible to determine $C(R)$, and since $C(R)$ is obviously a non-

increasing function of R bounded below by zero, it is natural to seek the one-sided limit of $C(R)$ as $R \rightarrow 1-0$. Thus, to each given function $z^2 p^*(z)$ there corresponds a universal constant $A = A(p^*)$ defined as

$$(5.16) \quad A = A(p^*) = \lim_{R \rightarrow 1-0} C(R)$$

which is finite, except when $z^2 p^*(z) \equiv \text{constant } p_0^*$ in which case $A = \infty$. In other words, A is the largest value of C for which $W'_C(r) > 0$ for all values of r in the interval $0 < r < 1$. To see this we note from (5.13) that $W'_C(r) > 0$ for all r in $0 < r < R < 1$ when $0 \leq C \leq C(R)$, and in particular $W'_C(r) > 0$ for all r in $0 < r < R < 1$ when $0 \leq C \leq A$. Since A is independent of R and R may be taken as near to 1 as we like, we have $W'_C(r) > 0$ for all r in $0 < r < 1$ for $0 \leq C \leq A$. Thus

$$(5.17) \quad W'_A(r) > 0 \quad \text{for all } r \text{ in } 0 < r < 1.$$

On the other hand, we have, from (5.13), $W'_C(r) \leq 0$ for $C(R) < C < C(R) + \delta$ for all small $\delta > 0$ for at least one value of r in $0 < r < 1$. It follows then that for small δ and R near enough to 1 we have, for all $\epsilon > 0$ arbitrarily small,

$$A \leq C(R) < A + \epsilon < C(R) + \delta$$

in which case

$$(5.18) \quad W'_C(r) \leq 0 \quad \text{for } C = A + \epsilon$$

for all small $\epsilon > 0$ for some r in $0 < r < 1$. Because of (5.17) and (5.18) we have shown that for each $\epsilon > 0$ there exists some $r = r(\epsilon)$ in $0 < r < 1$ for which

$$(5.19) \quad W'_A(r) > 0, \quad W'_{A+\epsilon}(r) \leq 0.$$

(5.17) and (5.19) show that A is the largest value of C for which $W'_C(r) > 0$ for all r in $0 < r < 1$.

We remark also that for every $\delta_1 > 0$, there exists a $\delta \leq \delta_1$ for which $C(R) = A + \delta$ for some R in $0 < R < 1$, in which case $W'_{A+\delta}(R) = 0$. If this were not so, since $C(R)$ is nonincreasing and $A = \lim_{R \rightarrow 1-0} C(R)$ we would have $C(R) = A$ for an interval $1 - \epsilon < R < 1$. In that case $W'_A(R) = 0$ for an $R < 1$. This contradicts the fact that $W'_A(r) > 0$ for all r in $0 < r < 1$ as we have shown above by (5.17). We conclude then that A is the largest value of C for which $|z| W'_C(|z|) / W_C(|z|) > 0$, when $z^2 p^*(z)$ is not a constant, and $|z| < 1$.

We shall presently give examples of functions $p^*(z)$ for which positive constants $A(p^*)$ are determined.

It is clear also that Theorem A may be restated with $A(p^*)$ replacing C , and inequalities (5.4), (5.5), and (5.6) then hold for $|z| < 1$ with $C = A(p^*)$.

6. The main theorem. Let us now define $F(z)$ as in (1.5)

$$(6.1) \quad F(z) = \{W(z)\}^{1/\alpha} = z + \dots$$

Similarly, we write

$$(6.2) \quad F_A(z) = \{W_A(z)\}^{1/\alpha^*} = z + \dots$$

Then both $F(z)$ and $F_A(z)$ are regular and single-valued in $|z| < 1$. Furthermore,

$$(6.3) \quad \Re \left\{ \alpha e^{i\gamma} \frac{zF'(z)}{F(z)} \right\} = \Re \left\{ e^{i\gamma} \frac{zW'(z)}{W(z)} \right\},$$

$$(6.4) \quad \alpha^* \Re \left\{ \frac{zF'_A(z)}{F_A(z)} \right\} = \Re \left\{ \frac{zW'_A(z)}{W_A(z)} \right\}.$$

By Theorem A we have

$$(6.5) \quad \Re \left\{ \alpha e^{i\gamma} \frac{zF'(z)}{F(z)} \right\} \geq \frac{|z| W'_A(|z|)}{W_A(|z|)} \cos \gamma \geq 0, \quad |z| < 1,$$

$$(6.6) \quad \Re \left\{ \frac{zF'_A(z)}{F_A(z)} \right\} \geq \frac{1}{\alpha^*} \frac{|z| W'_A(|z|)}{W_A(|z|)} > 0, \quad |z| < 1.$$

Thus, $F(z)$ is schlicht and spiral-like in $|z| < 1$ for $R\alpha \geq 1/2$. Furthermore, $F_A(z)$ is schlicht and star-like in $|z| < 1$ for $\alpha^* \geq 1/2$. Since equality signs hold in (6.6) when z is positive, and since we have seen that $W'_{A+\epsilon}(R) = 0$ for some R in $0 < R < 1$ and arbitrarily small but positive ϵ , we conclude that $F_{A+\epsilon}(z)$ is not schlicht no matter how small $\epsilon > 0$ is taken.

We shall show now that, if $A(p^*) > 0$ and $z^2 p^*(z)$ is not identically a constant, then the radius of univalence (defined to be the largest circle with center at the origin within which the function is both regular and schlicht) of $F_A(z)$ is precisely one. To begin with, let us suppose that $p^*(z)$ has a singularity on $|z| = 1$ and $A(p^*) > 0$. From the differential equation (6.10) below it follows that $W''_A(z)/W_A(z)$ also has a singularity on $|z| = 1$. Thus $W_A(z)$ either has a zero or a singularity on $|z| = 1$. In either case $F_A(z)$ cannot be both regular and schlicht in any circle containing the unit circle $|z| = 1$. In the second place, if $p^*(z)$ is regular on $|z| = 1$, then so is the function $W_A(z)$. In this case, assuming $A(p^*) > 0$ and $z^2 p^*(z)$ not a constant, we may take $R = 1$ in Theorem A, $A(p^*) = C(1)$, and $W'_A(1) = W'_{C(1)}(1) = 0$. Thus, in this second case, the derivative of $F_A(z)$ vanishes on the unit circle. In either of the two cases we conclude that the radius of univalence for $F_A(z)$ is one.

It seems desirable at this point to summarize our conclusions in the following theorem, the principal object of this paper.

THE MAIN THEOREM. *Let the nonconstant function*

$$(6.7) \quad z^2 p^*(z) = p_0^* + p_1^* z + \dots + p_n^* z^n + \dots$$

be regular for $|z| < 1$, real on the real axis and $p_0^ \leq 1/4$. Let*

$$(6.8) \quad \Re \{ z^2 p^*(z) \} \leq |z|^2 p^*(|z|) \quad \text{for } |z| < 1.$$

Let $A = A(p^*)$ be the universal constant associated with $p^*(z)$ as determined by (5.15) and (5.16). Let

$$(6.9) \quad W_A(z) = z^{\alpha^*} \sum_{n=0}^{\infty} a_n^* z^n, \quad a_0^* = 1, |z| < 1,$$

be the unique solution of

$$(6.10) \quad W'' + \left\{ A \left(p^*(z) - \frac{p_0^*}{z^2} \right) + \frac{p_0^*}{z^2} \right\} W = 0$$

corresponding to the larger root α^* of the indicial equation. Then the function

$$(6.11) \quad F_A(z) = \{W_A(z)\}^{1/\alpha^*} = z + \dots$$

is regular, single-valued, schlicht and star-like with respect to the origin in $|z| < 1$, and is not both regular and schlicht in any larger circle whenever $A(p^*) > 0$, and $z^2 p^*(z)$ is not a constant. For arbitrarily small and positive ϵ the function $F_{A+\epsilon}(z)$ is not schlicht in $|z| < 1$.

Let $z^2 p(z)$ be regular for $|z| < 1$, and γ a real constant ($|\gamma| \leq \pi/2$) for which in $|z| < 1$

$$(6.12) \quad \Re\{e^{i\gamma} z^2 p(z)\} \leq \cos \gamma \{A(|z|^2 p^*(|z|) - p_0^*) + p_0^*\}$$

where $A = A(p^*)$. Let

$$(6.13) \quad W(z) = z^{\alpha} \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1, |z| < 1,$$

be the unique solution of

$$(6.14) \quad W'' + p(z)W = 0$$

corresponding to the root α , with the larger real part, of the indicial equation. Then the function

$$(6.15) \quad F(z) = \{W(z)\}^{1/\alpha} = z + \dots$$

is regular, single-valued, schlicht and spiral-like in $|z| < 1$. The constant $A = A(p^*)$ is the largest possible one.

We remark that if $z^2 p^*(z)$ is a constant p_0^* , then $A(p^*) = \infty$ and, for all C , $W_C(z) = z^{\alpha^*}$, $2\alpha^* = 1 + (1 - 4p_0^*)^{1/2}$, in which case the function (6.11) is the trivial function z . However, in this case the right-hand side of (6.12) is indeterminate, as indeed is (6.10). If it should happen that for some functions $p(z)$ the real part of $\{e^{i\gamma} z^2 p(z)\}$ is bounded above by some constant K , then we may deduce from the theory of functions with bounded real part that

$$(6.16) \quad \Re\{e^{i\gamma} z^2 p(z)\} \leq \frac{2K|z|}{1+|z|} + \frac{1-|z|}{1+|z|} \Re(p_0 e^{i\gamma}), \quad |z| < 1.$$

In this case, rather than take $z^2 p^*(z)$ a constant in our theorem above we may take $z^2 p^*(z) = z(1+z)^{-1} + p_0^*$, and the value of $A(p^*)$ which goes with this choice. In this way an appropriate value for K is determined.

COROLLARY 1. *The main theorem holds in particular if (6.8) is replaced by the condition that $\{z^2 p^*(z) - p_0^*\}$ be convex in the direction of the imaginary axis for $|z| < 1$, and if (6.12) is replaced by the conditions that the function $\{z^2 p(z) - p_0\}$ be subordinate to the function $A(p^*)\{z^2 p^*(z) - p_0^*\}$ in $|z| < 1$ and that $\Re p_0 \leq p_0^*$.*

Since $\{z^2 p^*(z) - p_0^*\}$ is to be convex in the direction of the imaginary axis, and real on the real axis, it follows that the $\max_{|z|=r} \Re\{z^2 p^*(z)\}$ occurs for $z=r$ and (6.8) then holds. If also

$$(6.17) \quad \{z^2 p(z) - p_0\} << A(p^*)\{z^2 p^*(z) - p_0^*\},$$

and if $\Re p_0 \leq p_0^*$, we have

$$(6.18) \quad \Re\{z^2 p(z) - p_0\} \leq A(p^*)\{r^2 p^*(r) - p_0^*\}$$

and

$$(6.19) \quad \Re\{z^2 p(z)\} \leq A(p^*)\{r^2 p^*(r) - p_0^*\} + p_0^*.$$

Thus (6.12) holds for $\gamma=0$. This completes the proof of Corollary 1.

7. Illustrative examples. Since $z^2 p^*(z)$ was chosen to be not a constant in the main theorem, we shall take the next simplest case for our first illustration.

EXAMPLE 1. Let $z^2 p^*(z) = p_0^* + z$, $p_0^* \leq 1/4$. It will be convenient to write

$$p_0^* = \alpha^* - \alpha^{*2}, \quad \alpha^* \geq 1/2.$$

Equation (4.2) becomes

$$(7.1) \quad \frac{d^2 W}{dz^2} + \left(\frac{C}{z} + \frac{\alpha^* - \alpha^{*2}}{z^2} \right) W = 0.$$

The solution (4.5) of (7.1) is

$$(7.2) \quad W_C(z) = \Gamma(2\alpha^*) z^{\alpha^*} \sum_{n=0}^{\infty} \frac{(-Cz)^n}{n! \Gamma(n + 2\alpha^*)},$$

$$(7.3) \quad W_C(z) = \frac{\Gamma(2\alpha^*)}{C^{\alpha^*}} (Cz)^{1/2} J_{2\alpha^*-1}(2(Cz)^{1/2}).$$

The equation $W'_C(R) = 0$ leads to

$$(7.4) \quad J_{2\alpha^*-1}(2(CR)^{1/2}) + 2(CR)^{1/2} J'_{2\alpha^*-1}(2(CR)^{1/2}) = 0.$$

Thus

$$(7.5) \quad C(R) = \frac{X_1^2(\alpha^*)}{4R},$$

where $X_1 = X_1(\alpha^*)$ is the smallest positive root of the equation

$$(7.6) \quad J_{2\alpha^*-1}(X) + XJ'_{2\alpha^*-1}(X) = 0.$$

$$(7.7) \quad A = A(p^*) = \lim_{R \rightarrow 1-0} C(R) = \frac{X_1^2(\alpha^*)}{4}.$$

$$(7.8) \quad W_A(z) = \left(\frac{2}{X_1}\right)^{2\alpha^*-1} \Gamma(2\alpha^*) z^{1/2} J_{2\alpha^*-1}(X_1 z^{1/2}).$$

$$(7.9) \quad F_A(z) = \left[\left(\frac{2}{X_1}\right)^{2\alpha^*-1} \Gamma(2\alpha^*) z^{1/2} J_{2\alpha^*-1}(X_1 z^{1/2}) \right]^{1/\alpha^*} = z + \dots.$$

The function $F_A(z)$ of (7.9) is schlicht and star-like in $|z| \leq 1$, and its derivative vanishes at $z=1$. Thus the radius of univalence of $F_A(z)$ has the value 1.

If $\alpha^* = 1$ ($p_0^* = 0$), we have as a special case the result that the function

$$(7.10) \quad \phi(z) = \frac{2}{X_1} z^{1/2} J_1(X_1 z^{1/2}) = z + \dots,$$

where X_1 is the smallest positive zero of $J_0(X)$, $X_1 = 2.405 \dots$, is schlicht and star-like in $|z| \leq 1$, but is not schlicht in any larger circle. As a consequence we have the theorem

THEOREM 1. *Let $zp(z)$ be regular for $|z| < 1$, and*

$$\Re\{z^2 p(z)\} \leq \frac{X_1^2}{4} |z| \quad \text{for } |z| < 1,$$

where X_1 is the smallest positive zero of $J_0(X)$ ($X_1^2/4 = 1.4460 \dots$). Let

$$W = W(z) = z + a_2 z^2 + \dots + a_n z^n + \dots, \quad |z| < 1,$$

be the unique solution $W = W(z)$, $W(0) = 0$, $W'(0) = 1$, of the equation

$$W'' + p(z)W = 0.$$

Then $W = W(z)$ is schlicht and star-like in $|z| < 1$. The constant $X_1^2/4$ is a best possible one.

EXAMPLE 2. Let $z^2 p^*(z) = z^2$, $\alpha^* = 1$. Equation (4.2) becomes

$$(7.11) \quad \frac{d^2 W}{dz^2} + CW = 0.$$

The solution (4.5) of (7.11) is

$$(7.12) \quad W_C(z) = C^{-1/2} \sin(C^{1/2}z).$$

We find $W'_C(R) = 0$ for $C^{1/2}R = \pi/2$. Thus

$$(7.13) \quad C(R) = \frac{\pi^2}{4R^2}, \quad A(p^*) = \frac{\pi^2}{4}.$$

$$(7.14) \quad W_A(z) = \frac{2}{\pi} \sin\left(\frac{\pi}{2}z\right) = z + \dots.$$

$W_A(z)$ of (7.14) has a radius of univalence equal to 1 and its derivative vanishes at $z=1$. We then have the theorem

THEOREM 2. Let $zp^*(z)$ be regular in $|z| < 1$ and $\Re\{z^2p^*(z)\} \leq (\pi^2/4)|z|^2$ in $|z| < 1$. Then the unique solution $W = W(z)$, $W(0) = 0$, $W'(0) = 1$ of

$$W'' + p(z)W = 0$$

is schlicht and star-like for $|z| < 1$. The constant $\pi^2/4$ is a best possible one.

EXAMPLE 3. Let $z^2p^*(z) = z/(1+z)$, $\alpha^* = 1$. In this case $z^2p^*(z)$ is a convex function, real on the real axis. Thus for $|z| = r < 1$

$$(7.15) \quad \max_{|z|=r} \Re\{z^2p^*(z)\} = \frac{r}{1+r} = |z|^2p^*(|z|).$$

The solution $W_C(z)$, $W_C(0) = 0$, $W'_C(0) = 1$ of

$$(7.16) \quad W'' + \frac{C}{z(1+z)}W = 0$$

is

$$(7.17) \quad W_C(z) = \frac{-1}{C} \sum_{n=1}^{\infty} \prod_{k=1}^n \left\{ \frac{(2k-3)^2 - 1}{4} + C \right\} \frac{(-z)^n}{(n-1)!n!},$$

$$(7.18) \quad W_C(z) = zF\left(\frac{1 + (1-4C)^{1/2}}{2}, \frac{1 - (1-4C)^{1/2}}{2}; 2; -z\right)$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function

$$(7.19) \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)n!} z^n, \quad |z| < 1.$$

The equation $W'_C(R) = 0$ leads to

$$(7.20) \quad F(\alpha, \beta; 2; -R) - RF'(\alpha, \beta; 2; -R) = 0,$$

where

$$(7.21) \quad 2\alpha = 1 + (1-4C)^{1/2}, \quad 2\beta = 1 - (1-4C)^{1/2}, \quad \alpha + \beta = 1.$$

The equation (7.20) may be written as

$$(7.22) \quad (1 - \beta)F(\alpha, \beta; 2; -R) + \beta F(\alpha, \beta + 1; 2; -R) = 0.$$

The series for $F(\alpha, \beta; 2; -1)$ converges absolutely while the series for $F(\alpha, \beta + 1; 2; -1)$ converges conditionally for the values of α and β given in (7.21).

Using the integral representation

$$(7.23) \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt,$$

valid when $|z| < 1$, $\Re \gamma > \Re \beta > 0$, in (7.22) we find that the left side of equation (7.22) is positive for $0 < C \leq 1/4$. Letting

$$(7.24) \quad s = \frac{1}{2} (4C - 1)^{1/2} > 0,$$

$$C = s^2 + \frac{1}{4},$$

and equating to zero the real part of the integral representation of the left side of (7.22) we obtain, after considerable simplification, the equation

$$(7.25) \quad \int_0^1 \frac{\cos [s \log ((1-t)/(t+Rt^2))]}{[t(1-t)(1+Rt)]^{1/2}} dt = 0.$$

$$(7.26) \quad A(p^*) = \lim_{R \rightarrow -0} C(R) = \lim_{R \rightarrow 1} s^2(R) + \frac{1}{4} = s_1^2 + \frac{1}{4}$$

where s_1 is the smallest positive zero of the equation (7.25) when $R=1$. Putting $e^x = (1-t)(t+t^2)^{-1}$ in (7.25) we find that s_1 is the smallest positive zero of the function $\phi(s)$ defined as

$$(7.27) \quad \phi(s) = \int_0^\infty (3 + \cosh x)^{-1/2} \cos sx dx.$$

By considering the contour integral

$$(7.28) \quad \oint e^{(s+i/2)z} (e^{2z} + 6e^z + 1)^{-1/2} dz = 0,$$

where the contour is the rectangle with corners at $z=0$, R , $R+2\pi i$, $2\pi i$ and a slit, parallel to the real axis and joining the points πi and $2 \ln (2^{1/2}+1) + \pi i$, it is possible to show, after letting $R \rightarrow \infty$ and taking real parts, that

$$(7.29) \quad \phi(s) \equiv \operatorname{sech} s\pi \int_0^{2 \ln (2^{1/2}+1)} (3 - \cosh x)^{-1/2} \cos sx dx.$$

From (7.29) it can be shown that

$$\phi(s) > 0, \quad 0 \leq s \leq \frac{\pi}{4} [\ln(2^{1/2} + 1)]^{-1}, \quad (7.30)$$

$$\phi(s) < 0, \quad \frac{\pi}{2} [\ln(2^{1/2} + 1)]^{-1} < s < \frac{3\pi}{4} [\ln(2^{1/2} + 1)]^{-1}.$$

The existence of the zero s_1 of $\phi(s)$ follows from (7.30). It lies between 1.3 and 1.4. We omit the details of the proof of this statement.

If $zp(z)$ is regular in $|z| < 1$ and

$$(7.31) \quad \Re\{z^2 p(z)\} \leq A(p^*)/2 \quad \text{in } |z| < 1,$$

where $A(p^*) = s_1^2 + 1/4$, and s_1 is the smallest positive zero of $\phi(s)$, determined by (7.27) or (7.29), then for $|z| < 1$

$$(7.32) \quad \Re\{z^2 p(z)\} \leq A(p^*) \frac{|z|}{1 + |z|} = A(p^*) |z|^2 p^*(|z|),$$

and (6.12) is satisfied. From this we have the theorem:

THEOREM 3. *If $zp(z)$ is regular in $|z| < 1$, and if*

$$\Re\{z^2 p(z)\} \leq \frac{1}{2} s_1^2 + \frac{1}{8}, \quad |z| < 1,$$

where s_1 is the smallest positive zero of the function

$$\phi(s) = \int_0^\infty (3 + \cosh x)^{-1/2} \cos sxdx,$$

then the unique solution $W = W(z)$ of the form

$$W = z + \sum_{n=2}^\infty a_n z^n, \quad |z| < 1,$$

of the equation

$$\frac{d^2 W}{dz^2} + p(z)W = 0,$$

is schlicht and star-like in $|z| < 1$. The constant $s_1^2/2 + 1/8$ cannot be replaced by a larger one.

We remark that for $C = A(p^*) = s_1^2 + 1/4$ the hypergeometric function (multiplied by z) in (7.18) is schlicht in $|z| \leq 1$ and its derivative vanishes at $z = 1$. This solution corresponds to a choice of

$$(7.33) \quad z^2 p(z) = A(p^*) z^2 p^*(z) = A(p^*) z (1 + z)^{-1}$$

in which case $p_0=0$, $\alpha=1$.

If $p_0 \neq 0$ and if $z^2 p(z)$ is regular in $|z| < 1$, Theorem 3 could have been stated in a somewhat more general form provided we assume $\Re p_0 \leq 0$ and use (6.16) with $\gamma=0$. In this case $\alpha \neq 1$.

EXAMPLE 4. Let

$$(7.34) \quad z^2 p^*(z) = \sum_{n=0}^{\infty} p_n^* z^n, \quad p_0^* \leq \frac{1}{4},$$

be regular in $|z| < 1$ with $p_n^* \geq 0$, $n=1, 2, \dots$. Suppose $|z|=1$ is a natural boundary for $z^2 p^*(z)$. This is the situation if, for instance, the series has sufficiently large gaps. Since none of the coefficients is negative, the condition

$$(7.35) \quad \max_{|z|=r} \Re \{ z^2 p^*(z) \} = |z|^2 p^*(|z|)$$

of the main theorem is fulfilled. We then determine the constant $A=A(p^*)$ by (5.15) and (5.16). If $A > 0$ we see from (6.9) and (6.10) that the solution $F_A(z)$ of (6.11) corresponding to our choice of $z^2 p^*(z)$ in this example is schlicht and star-like in $|z| < 1$, and, moreover, has the unit circle as a natural boundary. Thus we have a device for constructing schlicht functions with natural boundaries whenever $A(p^*)$ can be determined in a constructive way, and provided it is not zero.

EXAMPLE 5. That $A=A(p^*)$ can sometimes be zero is shown by the following illustration. Let

$$(7.36) \quad z^2 p^*(z) = z^2(1-z^2)^{-2}.$$

Here $p_0^*=0$. The solution [3] corresponding to $\alpha^*=1$ of

$$(7.37) \quad W'' + \frac{C}{(1-z^2)^2} W = 0, \quad C \geq 0,$$

is

$$(7.38) \quad W_C(z) = (1-z^2)^{1/2} \cdot \frac{((1+z)/(1-z))^{\delta/2} - ((1-z)/(1+z))^{\delta/2}}{2\delta},$$

$$\delta = (1-4C)^{1/2} \neq 0.$$

$$(7.39) \quad W'_C(r) = [(\delta-r)(1+r)^\delta + (\delta+r)(1-r)^\delta] \div 2\delta(1-r^2)^{(1+\delta)/2}.$$

Let $y=y(\delta)$ be the numerator of (7.39): For values of r sufficiently close to, but less than, one and for $\epsilon > 0$ arbitrarily small we have

$$\begin{aligned} y &= 2r(1-r)^r > 0 & \text{when } \delta = r < 1, \\ y &= -\epsilon r(1+r)^{(1-\epsilon)r} + (2-\epsilon)r(1-r)^{(1-\epsilon)r} < 0 & \text{when } \delta = (1-\epsilon)r. \end{aligned}$$

Thus $y=0$ for at least one root $\delta=\delta_0$, $(1-\epsilon)r < \delta_0 < r$. As $r \rightarrow 1$, $\delta_0 \rightarrow 1$, since ϵ

may be taken arbitrarily small. In this case $C = C(r) \rightarrow 0$. Thus $A(p^*) = 0$, $W_A(z) \equiv z$.

Although $W_C(z)$ in (7.38) is indeterminate when $\delta = 0$ ($C = 1$), a limiting process gives

$$(7.40) \quad W_1(z) = \frac{1}{2} (1 - z^2)^{1/2} \log \left(\frac{1+z}{1-z} \right).$$

The derivative of $W_1(z)$ vanishes within the unit circle. Since $W_1(z)$ is therefore not schlicht in $|z| < 1$, it is sufficient to consider as we did only the range $0 \leq C < 1$ for $W_C(z)$ in (7.38) (δ real and positive).

8. Concluding remarks. Throughout this paper we have confined our investigation to the solution of

$$(8.1) \quad W'' + p(z)W = 0, \quad z^2 p(z) \text{ regular in } |z| < 1,$$

which corresponds to that root α of the indicial equation for solutions about the origin for which the real part of α is the larger (or, if the real parts are both equal, to a solution about the origin which does involve $\log z$). The reason for this is fairly obvious: the integrals in the Green's Transform (3.4) do not exist for $\Re \alpha < 1/2$.

This, however, poses the question as to whether our main theorem may not still have a counterpart for the other root β , if we assume $\Re \beta > 0$ and employ a modified method of proof. I am leaving this question open for further investigation, but point out here that the Green's transform may be rewritten so that the integrals exist for $\Re \beta > 0$. We multiply (8.1) by $z\bar{W}dz$ and integrate from 0 to z , $|z| < 1$. This gives

$$(8.2) \quad \int_0^z z\bar{W}(z)W''(z)dz + \int_0^z zp(z)|W(z)|^2dz = 0.$$

Integrating by parts, we obtain

$$(8.3) \quad \begin{aligned} |W(z)|^2 \frac{zW'(z)}{W(z)} &= \int_0^z |W'(z)|^2 z d\bar{z} + \int_0^z W'(z)\bar{W}(z)dz \\ &\quad - \int_0^z zp(z)|W(z)|^2 dz. \end{aligned}$$

If the path of integration from 0 to $z = re^{i\theta}$ is a straight line segment, $\theta = \text{constant}$, we have

$$(8.4) \quad \begin{aligned} |W(z)|^2 \Re \left\{ \frac{zW'(z)}{W(z)} \right\} &= \int_0^r |W'|^2 \rho d\rho + \int_0^r \Re \left\{ \frac{zW'}{W} \right\}_{|z|=\rho} |W|^2 \frac{d\rho}{\rho} \\ &\quad - \int_0^r \Re \{ z^2 p(z) \}_{|z|=\rho} |W|^2 \frac{d\rho}{\rho}. \end{aligned}$$

It is seen at once that in this modified form Green's transform involves integrals which exist for $\Re\beta > 0$. However, an additional term has been added to the formula which means that some further modifications of attack on the problem are necessary to obtain results for the case $\Re\beta > 0$ analogous to those found in this paper.

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